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On the construction of the Grothendieck fundamental group of a topos by paths

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Abstract

The purpose of this paper is to compare the construction of the Grothendieck fundamental group of a topos using locally constant sheaves, with the construction using paths given by Moerdijk and Wraith. Our discussion focuses on the Grothendieck fundamental group in the general case of an unpointed (possibly pointless) topos, as constructed by Bunge. Corresponding results for topoi with a chosen base-point are then easily derived. The main result states that the basic comparison map from the paths fundamental group to the (unpointed version of the) Grothendieck fundamental group is an equivalence, under assumptions of the "locally paths simply connected" sort, as for topological spaces. © 1997 Elsevier Science B.V.

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0. Introduction

The purpose of this paper is to compare the construction of the Grothendieck fundamental group of a topos using locally constant sheaves [4] with the construction using paths given in [18, 13]. Our discussion focuses on the Grothendieck fundamental group in the general case of an unpointed (possibly pointless) topos, as in [2]; corresponding results for topoi with a chosen base-point are then easily derived. The main result states that the basic comparison map from the paths fundamental group to the

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(unpointed version of the) Grothendieck fundamental group is an equivalence, under assumptions of the "locally path simply connected" sort, as for topological spaces.

1. Locally constant sheaves and the Grothendieck fundamental group

In this section we will briefly review Grothendieck's theory of the fundamental group [4, 5] of a pointed topos, and some of its refinements presented in [1, 15].

Throughout this section, as well as in the rest of the paper, $\mathscr{E}$ denotes a connected and locally connected topos over an arbitrary base topos $\mathscr{S}$; its structure map will be denoted by $\gamma : \mathscr{E} \to \mathscr{S}$. We will often use notation and terminology as if $\mathscr{S}$ were the category of sets, as usual.

Recall that an object of L of $\mathscr{E}$ is said to be *locally constant* if there exists an epimorphism $U \twoheadrightarrow 1$ in $\mathscr{E}$ and an isomorphism $U \times L \simeq U \times \gamma^*(S)$ over U, for some set S (i.e., some object of $\mathscr{S}$); the object U is then said to *trivialize* L. One also says that $\mathscr{E}/L \to \mathscr{E}$ is a *covering (projection) of* $\mathscr{E}$, or that L is a covering of $\mathscr{E}$. A geometric morphism $\mathscr{F} \to \mathscr{E}$ is called a covering projection if it is equivalent (over $\mathscr{E}$) to one such of the form $\mathscr{E}/L \to \mathscr{E}$. A locally constant sheaf L (i.e., an object of $\mathscr{E}$) is said to be *finite* if the set S above is finite. We denote by $LC(\mathscr{E})$, $FLC(\mathscr{E})$ and $SLC(\mathscr{E})$ the full subcategories of $\mathscr{E}$ consisting of locally constant objects, of finite locally constant objects, and of sums of locally constant objects, respectively.

For the case $\mathscr{S} = Set$, and under the assumption that $\mathscr{E}$ has a chosen base-point $p : Set \to \mathscr{E}$, Grothendieck's Galois theory [4] shows that the category $FLC(\mathscr{E})$ is equivalent to the category of finite continuous G-sets, by an equivalence which identifies $p^* : FLC(\mathscr{E}) \to Set$ with the forgetful functor, for a unique profinite group G. This G is then called the *profinite fundamental group of* $\mathscr{E}$ and denoted $\pi_1^{\mathrm{pf}}(\mathscr{E}, p)$.

The construction of the profinite fundamental group does not use the assumption that $\mathscr{E}$ is locally connected. Using this assumption, one can construct in an analogous way a localic group G such that the category $SLC(\mathscr{E})$ is equivalent to the category $\mathscr{B}G$ of continuous G-sets (relative to $\mathscr{S}$), again by an equivalence which identifies p^* with the forgetful functor. This time G is a prodiscrete localic group, denoted $\pi_1(\mathscr{E}, p)$. (This construction is given in [15].)

In the definition of locally constant sheaves (or objects), the trivializing object U can vary over all epimorphisms $U \twoheadrightarrow 1$. The topos $\mathscr{E}$ is said to be *locally simply connected* (l.s.c.) if there is *one* such epimorphism $U \twoheadrightarrow 1$ which trivializes *all* locally constant objects in $\mathscr{E}$. This case is discussed in detail in [1]. One has the following equivalent descriptions of local simple connectivity:

(i) $\mathscr{E}$ is locally simply connected,

(ii) $LC(\mathscr{E}) = SLC(\mathscr{E})$, i.e., locally constant objects are closed under sums,

(iii) the prodiscrete group $\pi_1(\mathscr{E}, p)$ is discrete.

In this case, there is a *universal cover* $\tilde{\mathscr{E}} \to \mathscr{E}$, i.e., $\tilde{\mathscr{E}} = \mathscr{E}/\tilde{L}$ for a locally constant object $\tilde{L}$, which has the property that $\pi_1(\mathscr{E}, p) = Aut(\tilde{L})$. Moreover, $\tilde{L}$ is universal in the usual sense that every connected object from $LC(\mathscr{E})$ is a quotient of $\tilde{L}$.

2. The coverings fundamental group of a topos in the absence of a point

We review an alternative construction of $SLC(\mathscr{E})$ along the lines of [2], for the case of a connected, locally connected topos $\mathscr{E}$ over a base topos $\mathscr{S}$. In this case, there is no chosen basepoint of $\mathscr{E}$, and $SLC(\mathscr{E})$ is equivalent to the category $\mathscr{B}G$ of continuous G-sets (relative to $\mathscr{S}$), with G a localic groupoid, denoted $\pi_1(\mathscr{E})$. It follows readily from the construction that $\pi_1(\mathscr{E})$ represents $H^1(\mathscr{E}, -)$.

For any epimorphism $U \twoheadrightarrow 1$, denote by $\mathscr{G}_U$ the pushout topos

$$\begin{array}{ccc} \mathscr{E}/U & \xrightarrow{\varphi_U} & \mathscr{E} \\ {\scriptstyle \rho_U}\downarrow & & \downarrow{\scriptstyle \sigma_U} \\ \mathscr{S}/\gamma_! U & \xrightarrow[p_U]{} & \mathscr{G}_U \end{array}$$

where all the mappings are the canonical ones, and $\gamma_!$ denotes the left adjoint of γ^*, which exists because γ is assumed to be locally connected. Thus, $\gamma_!(U)$ is the set of connected components of U (or of $\mathscr{E}/U$). We now sketch a proof of the main properties of $\mathscr{G}_U$, which then identifies $\mathscr{G}_U$ with the full subcategory of $\mathscr{E}$ consisting of the locally constant objects trivialized by U.

Proposition 2.1. *The topos $\mathscr{G}_U$ is connected and atomic, and the map $\sigma_U : \mathscr{E} \to \mathscr{G}_U$ is connected.*

Proof. Using [2, Lemma 2.3] we see that since φ_U and ρ_U are locally connected maps, so are p_U and σ_U; moreover, σ_U is connected since ρ_U is. Using then the properties of totally disconnected maps from [2, Section 1], we observe that p_U is totally disconnected and, since it is also locally connected, it must be a local homeomorphism (a slice). Using that p_U is of effective descent, it follows that $\gamma_!(U)$ is the set of objects of a discrete groupoid G_U with the property that $\mathscr{G}_U = \mathscr{B}G_U$. In particular, $\mathscr{G}_U$ is a connected, atomic topos. □

The construction of $\mathscr{G}_U$ is evidently functorial in U. Thus, for epimorphisms $U \twoheadrightarrow 1$, $V \twoheadrightarrow 1$, a map $U \to V$ yields a topos map $\mathscr{G}_U \to \mathscr{G}_V$, induced by a groupoid homomorphism $h_{UV} : G_U \to G_V$, such that the diagram

$$\begin{array}{ccc} & \mathscr{E} & \\ {\scriptstyle \sigma_U}\swarrow & & \searrow{\scriptstyle \sigma_V} \\ \mathscr{G}_U & \longrightarrow & \mathscr{G}_V \end{array}$$

commutes. Since σ_U, σ_V are connected so is the map $\mathscr{G}_U \to \mathscr{G}_V$. From this it follows that h_{UV} is a full and essentially surjective functor between groupoids. This implies that each $\mathscr{G}_U \to \mathscr{G}_V$ is in fact connected and atomic.

Now consider a small cofinal system in the category of all epimorphisms $U \twoheadrightarrow 1$, and form the inverse limit

$$\mathscr{G} = \varprojlim \mathscr{G}_U.$$

By [11], $\mathscr{G}$ is again connected and atomic. As a category, $\mathscr{G} = SLC(\mathscr{E})$, with $\sigma : \mathscr{E} \to SLC(\mathscr{E})$ induced by the σ_U, a map whose inverse image is simply the inclusion functor. We summarize all this as follows.

Theorem 2.2. *The topos $\mathscr{G} = SLC(\mathscr{E})$ is connected and atomic, and the map $\sigma : \mathscr{E} \to \mathscr{G}$ is connected and locally connected. Furthermore, $\mathscr{G} = \mathscr{B}G$, where G is the localic groupoid obtained as the (pseudo-, or lax-) limit $G = \varprojlim G_U$. The groupoid G classifies torsors, in the sense that for any group K in $\mathscr{S}$, there is a natural isomorphism $Hom(G,K) \cong H^1(\mathscr{E},K)$ (see [2] for details).*

Note that, in the above, G may be chosen to be etale complete and this determines it uniquely up to equivalence. We call G *the fundamental localic groupoid of* $\mathscr{E}$ and denote it by $\pi_1(\mathscr{E})$. An alternative definition of $\pi_1(\mathscr{E})$ is given in [9]. When $\mathscr{E}$ has a chosen basepoint $p : \mathscr{S} \to \mathscr{E}$, there is induced a point $\tilde{p}$ of G, and then $\pi_1(\mathscr{E}, p)$ is the vertex group $G_{\tilde{p}}$. The isomorphism $Hom(\pi_1(\mathscr{E}),K) \cong H^1(\mathscr{E},K)$ generalizes the isomorphism $Hom(\pi_1(\mathscr{E}, p),K) \cong H^1(\mathscr{E},K)$ given in [15] for the pointed case.

3. The paths fundamental group of a topos

As before, $\mathscr{E}$ will be a connected and locally connected topos over a fixed base topos $\mathscr{S}$. For any locale X in $\mathscr{S}$, we will also write X for the associated topos of sheaves; this again is a topos over $\mathscr{S}$. In particular, we will consider the unit interval I in $\mathscr{S}$, regarded as a locale, and the paths topos $\mathscr{E}^I$, constructed as the exponential of $\mathscr{E}$ by (the topos of sheaves on) I (cf. [6]). We recall from [18] the following result, which states that any connected and locally connected topos is "path connected".

Theorem 3.1. *The map $\varepsilon : \mathscr{E}^I \to \mathscr{E} \times \mathscr{E}$, defined by evaluation at the endpoints, is an open surjection.*

The interval I is part of the "standard" cosimplicial locale

$$1 = \Delta_0 \rightrightarrows I = \Delta_1 \;\substack{\longrightarrow\\[-2pt] \longrightarrow\\[-2pt] \longrightarrow}\; \Delta = \Delta_2 \cdots,$$

and by exponentiating one obtains a simplicial topos

$$\cdots \mathscr{E}^{\Delta} \;\substack{\longrightarrow\\[-2pt] \longrightarrow\\[-2pt] \longrightarrow}\; \mathscr{E}^I \;\underset{\varepsilon_0}{\overset{\varepsilon_1}{\rightrightarrows}}\; \mathscr{E}. \tag{1}$$

In [13], a topos $\Pi(\mathscr{E})$ is defined as the descent topos of the simplicial topos (1). Thus, $\Pi(\mathscr{E})$ is the topos of objects A of $\mathscr{E}$, equipped with "descent data" $\varepsilon_0^* A \simeq \varepsilon_1^* A$

satisfying the usual cocycle condition (in $\mathscr{E}^{\Delta}$). The forgetful functor $\Pi(\mathscr{E}) \to \mathscr{E}$ is part of a geometric morphism $q : \mathscr{E} \to \Pi(\mathscr{E})$. In [13], the following result is shown as an application of Theorem 3.1. We sketch the proof here.

Proposition 3.2. *The topos $\Pi(\mathscr{E})$ is connected and atomic, and the map $q : \mathscr{E} \to \Pi(\mathscr{E})$ is an open surjection.*

Proof. Since $\mathscr{E}$ is connected and locally connected, $\gamma : \mathscr{E} \to \mathscr{S}$ is an open surjection; moreover, by Theorem 3.1, so is $\mathscr{E}^I \to \mathscr{E} \times \mathscr{E}$. These imply (using [13]) that $q : \mathscr{E} \to \Pi(\mathscr{E})$ is an open surjection. To show that $\Pi(\mathscr{E})$ is atomic, consider the square

$$\begin{array}{ccc} \mathscr{E}^I & \xrightarrow{q\varepsilon_0} & \Pi(\mathscr{E}) \\ {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle \mathrm{diag}} \\ \mathscr{E} \times_{\mathscr{S}} \mathscr{E} & \xrightarrow[q \times q]{} & \Pi(\mathscr{E}) \times_{\mathscr{S}} \Pi(\mathscr{E}) \end{array}$$

Since $q\varepsilon_0$ and $(q \times q)\varepsilon$ are open surjections, so is *diag*. But (cf. [8]) any topos $\mathscr{E}$ with open structure map $\mathscr{E} \to \mathscr{S}$ and open diagonal $\mathscr{E} \to \mathscr{E} \times_{\mathscr{S}} \mathscr{E}$ is atomic. Finally, since $\mathscr{E}$ is assumed connected and g is an open surjection, $\Pi(\mathscr{E})$ is connected. □

When $\mathscr{E}$ has a chosen basepoint $p : \mathscr{S} \to \mathscr{E}$, it follows from Proposition 3.2 that $\Pi(\mathscr{E})$ is equivalent to the topos $\mathscr{B}G$ of continuous G-sets, by an equivalence which identifies the canonical point of $\mathscr{B}G$ with $qp : \mathscr{S} \to \Pi(\mathscr{E})$, for a localic group G (see [8]). Moreover, this group G can be chosen to be étale-complete, and this then determines it uniquely up to isomorphism (cf. [13]). We will call this unique group G the paths fundamental group of $(\mathscr{E}, p)$ and denote it by $\pi_1^{\mathrm{paths}}(\mathscr{E}, p)$. As stated at the beginning of the paper, it is the purpose here to compare $\Pi(\mathscr{E})$ with $SLC(\mathscr{E})$ and, consequently, when $\mathscr{E}$ has a chosen basepoint p, to compare $\pi_1^{\mathrm{paths}}(\mathscr{E}, p)$ with $\pi_1(\mathscr{E}, p)$. We begin, in the next section, with the (easy) observation that there is always a natural comparison map.

4. The comparison map

As before, $\mathscr{E}$ is a connected and locally connected topos over $\mathscr{S}$, with associated morphisms $\sigma : \mathscr{E} \to SLC(\mathscr{E})$ and $q : \mathscr{E} \to \Pi(\mathscr{E})$ constructed, respectively, in Sections 2 and 3. Recall that σ is connected and locally connected, while q is an open surjection. Recall also that $SLC(\mathscr{E})$ was constructed, in the unpointed case, as a limit topos, while $\Pi(\mathscr{E})$ was defined as a descent topos.

Proposition 4.1. *Up to isomorphism, there exists a unique geometric morphism $\varphi : \Pi(\mathscr{E}) \to SLC(\mathscr{E})$, such that $\varphi q \cong \sigma$. It follows that φ is connected and locally connected.*

Proof. The proof is a direct consequence of the simple fact that paths act uniquely on locally constant objects, and this can be proved much as in topology. Thus, to define the inverse image functor φ^*, let L be any locally constant object in $\mathscr{E}$. We aim to show that L has a natural action by paths, of the form of an isomorphism $\theta : \varepsilon_0^* L \to \varepsilon_1^* L$ satisfying the cocycle condition, as required in the definition of $\Pi(\mathscr{E})$.

For this, it is of course enough to prove that for any topos $\mathscr{F}$ and any morphism $f : \mathscr{F} \to \mathscr{E}^I$, there is an action $\theta_f : f^*\varepsilon_0^*(L) \to f^*\varepsilon_1^*(L)$, natural in f. Since L is locally constant in $\mathscr{E}$, its pullback $\hat{f}^*(L)$ along the transposed map $\hat{f} : \mathscr{F} \times_{\mathscr{S}} I \to \mathscr{E}$ is locally constant in $\mathscr{F} \times_{\mathscr{S}} I$. The latter is the topos of internal sheaves on the unit interval in $\mathscr{F}$. Since the unit interval is simply connected while $\hat{f}^*(L)$ is locally constant, it follows that $\hat{f}^*(L)$ is internally constant as a sheaf on I inside $\mathscr{F}$. In other words, for the projection $\pi : \mathscr{F} \times_{\mathscr{S}} I \to \mathscr{F}$, the counit $\pi^*\pi_*\hat{f}^*(L) \to \hat{f}^*(L)$ is an isomorphism.

Consider now the inclusion of the endpoints i_0 and i_1,

$$\mathscr{F} \times_{\mathscr{S}} I \;\underset{i_1}{\overset{i_0}{\leftleftarrows}}\!\!\xrightarrow{\ \pi\ } \mathscr{F},$$

with the associated canonical isomorphisms $\pi i_0 \cong \mathrm{id} \cong \pi i_1$ (which are natural in $\mathscr{F}$). The required isomorphism $\theta_f : f^*\varepsilon_0^* L \to f^*\varepsilon_1^* L$ can be defined in terms of canonical isomorphisms, as

$$\begin{array}{ccccc}
f^*\varepsilon_0^* L & \xrightarrow{\sim} & i_0^*\hat{f}^*L & \xrightarrow{\sim} & i_0^*\pi^*\pi_*\hat{f}^*L \\
\Big\downarrow{\scriptstyle \theta_f} & & & & \Big\downarrow{\wr} \\
 & & & & \pi_*\hat{f}^*L \\
 & & & & \Big\downarrow{\wr} \\
f^*\varepsilon_1^* L & \xrightarrow{\sim} & i_1^*\hat{f}^*L & \xrightarrow{\sim} & i_1^*\pi^*\pi_*\hat{f}^*L.
\end{array}$$

One readily verifies that θ_f, thus defined, is natural in f. It satisfies the appropriate cocycle condition, by similar considerations using the simple connectivity of Δ. This, then, defines a functor $\varphi^* : SLC(\mathscr{E}) \to \Pi(\mathscr{E})$. Since q^* is the forgetful functor, there is an obvious isomorphism between $q^*\varphi^*$ and the inclusion functor σ^*. In particular, $q^*\varphi^*$ preserves colimits and finite limits, and hence so does φ^* because q^* is faithful. Thus, φ^* is the inverse image of a geometric morphism φ, as claimed. The uniqueness of φ is due to the fact that the action θ_f on locally constant objects is unique: indeed, exactly as in topology, there is only one such action which satisfies the cocycle condition. Finally, since q is an open surjection, it follows from [13] that φ must be connected and locally connected since σ is. This proves the proposition. □

Remark 4.2. Suppose that $\mathscr{E}$ has a chosen basepoint p, so that there are equivalences of pointed topoi $SLC(\mathscr{E}) \cong \mathscr{B}\pi_1(\mathscr{E},p)$ and $\Pi(\mathscr{E}) \cong \mathscr{B}\pi_1^{\text{paths}}(\mathscr{E},p)$. The map φ constructed above commutes with the basepoints, and hence must be induced by a continuous homomorphism $\pi_1^{\text{paths}}(\mathscr{E},p) \to \pi_1(\mathscr{E},p)$ between (étale-complete) localic groups. The fact that φ is connected and locally connected implies that the image of this homomorphism is a dense subgroup of the prodiscrete group $\pi_1(\mathscr{E},p)$. (We will make no further use of this last observation.)

5. Path-simply connected topoi

As before, $\mathscr{E}$ is a fixed connected and locally connected topos over $\mathscr{S}$. Consider the localic 2-simplex Δ in the base topos $\mathscr{S}$, together with its boundary $\partial\Delta$. Identifying these locales with their topoi of sheaves, we define a topos $\mathscr{E}$ to be *path simply connected* (p.s.c.) if the canonical "restriction" map of exponential topoi

$$r : \mathscr{E}^{\Delta} \to \mathscr{E}^{\partial\Delta}$$

is a stable surjection.

Proposition 5.1. *If $\mathscr{E}$ is p.s.c., then $\Pi(\mathscr{E}) \cong \mathscr{S}$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc}
\mathscr{E}^{\partial\Delta} & \xrightarrow{\sim} & \mathscr{E}^I \times_{\mathscr{E}} \mathscr{E}^I & & & & \\
{\scriptstyle r}\uparrow & & {\scriptstyle p_0}\downarrow\downarrow{\scriptstyle p_1} & & & & \\
\mathscr{E}^\Delta & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathscr{E}^I & \underset{\varepsilon_1}{\overset{\varepsilon_0}{\rightrightarrows}} & \mathscr{E} & \xrightarrow{q} & \Pi(\mathscr{E}) \\
{\scriptstyle h}\downarrow & & {\scriptstyle \varepsilon}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \delta} \\
\mathscr{E}\times_{\mathscr{S}}\mathscr{E}\times_{\mathscr{S}}\mathscr{E} & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & \mathscr{E}\times_{\mathscr{S}}\mathscr{E} & \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} & \mathscr{E} & \xrightarrow{\gamma} & \mathscr{S}.
\end{array}$$

Here the two rows are descent diagrams, the upper one by definition of $\Pi(\mathscr{E})$ and the lower one since $\gamma : \mathscr{E} \to \mathscr{S}$ is connected and locally connected, hence an effective descent map. Observe that the assumptions on $\mathscr{E}$ imply that the map h is a surjection, since r and ε are (cf. Theorem 3.1).

Recall that the topos $\Pi(\mathscr{E})$ is connected, so that δ^* embeds $\mathscr{S}$ as a full subcategory of $\Pi(\mathscr{E})$. To show that the map δ is an equivalence of topoi, it thus suffices to prove that every object of $\Pi(\mathscr{E})$ is contained in the image of the functor δ^*. To this end, we prove that every object (X,θ) of $\Pi(\mathscr{E})$ carries canonical descent data for the bottom row. More precisely, we show that the descent data $\theta : \varepsilon_0^* X \to \varepsilon_1^* X$ descends along ε to a map $\rho : \pi_0^* X \to \pi_1^* X$. For such a ρ with $\varepsilon^*(\rho) = \theta$, the cocycle condition then follows immediately from the one for θ, by surjectivity of h.

To prove that θ descends along ε, it suffices to prove that θ is compatible with descent data for the pullback of ε along itself (one of the columns in the diagram); in other words, it suffices to prove that the square

$$\begin{array}{ccc} p_0^* \varepsilon_0^* X & \xrightarrow{p_0^*\theta} & p_0^* \varepsilon_1^* X \\ {\scriptstyle \mu_0 X}\big\downarrow & & \big\downarrow{\scriptstyle \mu_1 X} \\ p_1^* \varepsilon_0^* X & \xrightarrow[p_1^*\theta]{} & p_1^* \varepsilon_1^* X \end{array} \qquad (2)$$

commutes, where μ_i is the canonical isomorphism given by

$$\varepsilon_i p_0 \cong \pi_i \varepsilon p_0 \cong \pi_i \varepsilon p_1 \cong \varepsilon_i p_1.$$

But, under the equivalence $\mathscr{E}^{\Delta} \cong \mathscr{E}^{I} \times_{(\mathscr{E} \times_{\mathscr{S}} \mathscr{E})} \mathscr{E}^{I}$, this square (2) pulls back along f to the square expressing (a particular instance of) the cocycle condition for θ. Since f^* is faithful, (2) thus commutes. □

Recall that a topos $\mathscr{E}$ is said to be *simply connected* if every locally constant object in $\mathscr{E}$ in constant, i.e., if $SLC(\mathscr{E}) \cong \mathscr{S}$.

Corollary 5.2. *Every p.s.c. topos $\mathscr{E}$ is simply connected.*

Proof. Consider the comparison map φ of Proposition 4.1. Since $P\Pi(\mathscr{E}) \cong \mathscr{S}$ by Proposition 5.1, one has for any locally constant object L of $\mathscr{E}$ that $\sigma^*(L) \cong q^*\varphi^*(L)$ is a constant object in $\mathscr{E}$. □

6. Unique path-lifting

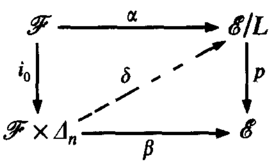
The following elementary lemma is the topos theoretic analogue of the familiar "unique path lifting" for covering projections in topology. Here Δ_n denotes the standard n-simplex (as a locale in the base topos $\mathscr{S}$), and $\varepsilon_0 : \mathscr{E}^{\Delta_n} \to \mathscr{E}$ is the evaluation at a vertex $v_0 : 1 \to \Delta_n$.

Lemma 6.1. *Let L be a locally constant object in $\mathscr{E}$. Then the square*

$$\begin{array}{ccc} (\mathscr{E}/L)^{\Delta_n} & \longrightarrow & \mathscr{E}^{\Delta_n} \\ {\scriptstyle \varepsilon_0}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon_0} \\ \mathscr{E}/L & \longrightarrow & \mathscr{E} \end{array}$$

is a pullback of topoi.

Proof. The required universal mapping property of the pullback translates by exponential transposition into the condition that any commutative (up to isomorphism) square below has a unique (up to isomorphism) diagonal fill-in δ, as indicated below



Here i_0 is the map corresponding to the vertex v_0, and p is the canonical map. Write $M = \beta^*(L)$. Then, internally in $\mathscr{F}$, M is a locally constant sheaf on Δ_n, and hence is (internally) constant, since Δ_n is contractible. The map α corresponds to a point a in the stalk M_{v_0} over the vertex v_0, internally in $\mathscr{F}$. Since M is constant, there is a unique section $d : \Delta^n \to M$ through this point, again internally in $\mathscr{F}$. Translating back to the external world, d corresponds to the required diagonal δ. □

For the following proposition, recall from Proposition 4.1 that the locally constant object L carries an action by paths, hence defines an object $\varphi^*(L)$ in $\Pi(\mathscr{E})$.

Proposition 6.2. *For L and $\mathscr{E}$ as above, there is a natural equivalence of topoi*

$$\Pi(\mathscr{E})/\varphi^*L \cong \Pi(\mathscr{E}/L).$$

Proof. The topos $\Pi(\mathscr{E})$ is defined by the descent diagram

$$\mathscr{E}^{\Delta} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \mathscr{E}^{I} \underset{\varepsilon_1}{\overset{\varepsilon_0}{\rightrightarrows}} \mathscr{E} \xrightarrow{q} \Pi(\mathscr{E}).$$

Slicing by φ^*L and using the fact that $q^*\varphi^*L \cong L$, one obtains a descent diagram

$$\mathscr{E}^{\Delta}/L \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} \mathscr{E}^{I}/L \rightrightarrows \mathscr{E}/L \longrightarrow \Pi(\mathscr{E})/\varphi^*L$$

(where the objects in $\mathscr{E}^{\Delta}$ and in $\mathscr{E}^{I}$ corresponding to L are again simply denoted L). By the preceding lemma, the latter diagram is equivalent to the diagram

$$(\mathscr{E}/L)^{\Delta} \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} (\mathscr{E}/L)^{I} \rightrightarrows \mathscr{E}/L \longrightarrow \Pi(\mathscr{E})/\varphi^*L.$$

But, by definition, $\Pi(\mathscr{E}/L)$ is the descent topos for this diagram; thus

$$\Pi(\mathscr{E}/L) \cong \Pi(\mathscr{E})/\varphi^*L. \qquad \square$$

7. The first equivalence theorem

For a connected and locally connected topos $\mathscr{E}$, we shall now prove a first result concerning the equivalence between $\Pi(\mathscr{E})$ and $SLC(\mathscr{E})$, stated in Proposition 7.1

below. This result is of course well-known for topological spaces, and was proved for locales and localic topoi in [3], in the unpointed case.

Recall from Section 1 that a locally simply connected topos $\mathscr{E}$ has a universal covering $u : \tilde{\mathscr{E}} \to \mathscr{E}$. This universal covering trivializes all locally constant objects in $\mathscr{E}$, and hence is simply connected, i.e., $SLC(\tilde{\mathscr{E}}) \cong \mathscr{S}$. We make the stronger assumption that $\Pi(\tilde{\mathscr{E}}) \cong \mathscr{S}$, as in the case when $\tilde{\mathscr{E}}$ is path simply connected (cf. Proposition 5.1).

Proposition 7.1. *Let $\mathscr{E}$ be a locally simply connected topos, and assume that its universal cover $u : \tilde{\mathscr{E}} \to \mathscr{E}$ has the property that $\Pi(\tilde{\mathscr{E}}) \cong \mathscr{S}$.* (This holds if $\tilde{\mathscr{E}}$ is path simply connected.) *Then the comparison map $\varphi : \Pi(\mathscr{E}) \to SLC(\mathscr{E})$ is an equivalence.*

Proof. Consider the diagram (cf. Proposition 4.1)

$$\begin{array}{ccc} & \mathscr{E} & \\ {}^{q}\swarrow & & \searrow^{\sigma} \\ \Pi(\mathscr{E}) & \xrightarrow[\varphi]{} & SLC(\mathscr{E}). \end{array}$$

Since φ is connected, φ^* embeds $SLC(\mathscr{E})$ as a full subcategory of $\Pi(\mathscr{E})$. Moreover, by unicity of φ, the action of paths on locally constant objects is unique. To prove that φ is an equivalence, it thus suffices to show that for every object (X, θ) of $\Pi(\mathscr{E})$, the object $X = q^*(X, \theta)$ in $\mathscr{E}$ is locally constant. To this end, consider the universal cover $u : \tilde{\mathscr{E}} \to \mathscr{E}$, and the associated map $\Pi(u) : \Pi(\tilde{\mathscr{E}}) \to \Pi(\mathscr{E})$. By the assumption on $\tilde{\mathscr{E}}$, the object $\Pi(u)^*(X, \theta)$ is constant, i.e., $u^*(X)$ is a constant object in $\tilde{\mathscr{E}}$. Thus, X is locally constant in $\mathscr{E}$, as required. □

Remark 7.2. It follows from the above, using Theorem 2.2, that $\Pi(\mathscr{E}) = \mathscr{B}(\pi_1(\mathscr{E}))$, with $\pi_1(\mathscr{E})$ a prodiscrete localic groupoid that classifies torsors, even if no definition of a corresponding $\pi_1^{\text{paths}}(\mathscr{E})$ is available in the unpointed case. For a chosen base-point $p : \mathscr{S} \to \mathscr{E}$, the above yields an isomorphism of localic groups,

$$\pi_1^{\text{paths}}(\mathscr{E}, p) \cong \pi_1(\mathscr{E}, p).$$

In particular, the paths fundamental group is discrete, since $\pi_1(\mathscr{E}, p)$ is, for any l.s.c. topos $\mathscr{E}$.

8. Locally path-simply connected topoi and the main comparison theorem

A connected and locally connected topos $\mathscr{E}$ is said to be *locally path-simply connected* (l.p.s.c.) if $\mathscr{E}$ has a generating system $\{C_i\}$ consisting of connected objects C_i with the property that each topos $\mathscr{E}/C_i$ is p.s.c. In particular, by Proposition 5.1, $\Pi(\mathscr{E}/C_i) \cong \mathscr{S}$.

Theorem 8.1. *For any l.p.s.c. topos $\mathscr{E}$, the comparison map $\varphi : \Pi(\mathscr{E}) \to SLC(\mathscr{E})$ is an equivalence.*

This theorem follows from Proposition 7.1 and the following two lemmas.

Lemma 8.2. *Any l.p.s.c. topos $\mathscr{E}$ is l.s.c.*

Proof. Let $U = C_i \twoheadrightarrow 1$ be a sum of generators which covers 1. We claim that U trivializes any locally constant object L of $\mathscr{E}$. For such an L, consider its pullback $L|C_i = (L \times C_i \to C_i)$ in $\mathscr{E}/C_i$. This object $L|C_i$ is locally constant, hence constant by the assumption on C_i and Corollary 5.2 (applied to $\mathscr{E}/C_i$). Thus, $L|C_i \cong (S_i \times C_i \to C_i)$ for some set S_i. For two different indices i and j, the sets S_i and S_j are isomorphic. Indeed, there is an isomorphism $S_i \times C_i \times C_j \cong L|(C_i \times C_j) \cong S_j \times C_i \times C_j$, over $C_i \times C_j$. Pulling back along a connected component $D \subseteq C_i \times C_j$, one obtains an isomorphism $S_i \times D \cong S_j \times D$ over D. Thus, $S_i \cong S_j$. This shows that for each i one can take the same set S, so that U trivializes L, as claimed. □

Lemma 8.2 implies that $\mathscr{E}$ has a universal cover $\tilde{\mathscr{E}}$ for which the following holds.

Lemma 8.3. *For any l.p.s.c. topos $\mathscr{E}$, its universal cover $\tilde{\mathscr{E}}$ has the property that $\Pi(\tilde{\mathscr{E}}) \cong \mathscr{S}$.*

Proof. Write $\tilde{\mathscr{E}} = \mathscr{E}/L$, so that $\Pi(\tilde{\mathscr{E}}) = \Pi(\mathscr{E})/L$ by Proposition 6.2 (we identify L and $\varphi^* L$ here). We have to show that every object $(X \to L)$ of $\Pi(\mathscr{E})/L$ is constant, as an object of $\mathscr{E}/L$. Since L defines the universal cover, it suffices to show that any such $(X \to L)$ is locally constant in $\mathscr{E}/L$. Cover L by sections $s_i : C_i \to L$ from simply connected generators. For each such section, $s_i^*(X \to L)$ is an object of $\Pi(\mathscr{E}/C_i)$, hence is constant there (cf. Proposition 5.1). It follows that $(X \to L)$ is locally constant, as required. □

Proof of Theorem 8.1. The conditions reduce, using Lemmas 8.2 and 8.3, to those of Proposition 7.1. It follows that $\varphi : \Pi(\mathscr{E}) \to SLC(\mathscr{E})$ is indeed an equivalence. □

Corollary 8.4. *Let $\mathscr{E}$ be a l.p.s.c. topos. Then $\Pi(\mathscr{E})$ is the classifying topos of a prodiscrete localic groupoid $\pi_1(\mathscr{E})$ that represents $H^1(\mathscr{E}, -)$:* Groups$(\mathscr{S}) \to \mathscr{S}$. *Further, if $\mathscr{E}$ has a chosen basepoint, then the canonical map $\pi_1^{\text{paths}}(\mathscr{E}, p) \to \pi_1(\mathscr{E}, p)$ is an isomorphism.*

9. The groupoid of paths

As before, $\mathscr{E}$ is a connected and locally connected topos over a base topos $\mathscr{S}$. An alternative construction of the fundamental groupoid of a topos by means of paths was proposed in [18], under the assumption that the "evaluation at the endpoints" map

$$\varepsilon : \mathscr{E}^I \to \mathscr{E} \times \mathscr{E}$$

is locally connected. When this is the case, let

$$\mathscr{E}^I \xrightarrow{\hat{\varepsilon}} \mathscr{H} \xrightarrow{\delta} \mathscr{E} \times \mathscr{E}$$

be the unique decomposition of ε into a connected and locally connected map $\hat{\varepsilon}$ followed by a local homeomorphism (a slice) δ. Thus, $\mathscr{H}$ is the topos of connected components of $\mathscr{E}^I$ as an $(\mathscr{E} \times \mathscr{E})$-topos. The two maps $\delta_0, \delta_1 : \mathscr{H} \rightrightarrows \mathscr{E}$ defined by δ are part of a groupoid topos

$$\mathscr{H} \times_{\varepsilon} \mathscr{H} \rightrightarrows\!\!\!\!\rightarrow \mathscr{H} \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} \mathscr{E} \tag{3}$$

with composition defined in the evident way using composition of paths. The "fundamental group" of $\mathscr{E}$ was defined in [18] as the descent topos $\mathscr{P}(\mathscr{E})$ of the groupoid topos (3).

Observe that there is an evident comparison map of simplicial topoi relating the simplicial topos (1) which defines $\Pi(\mathscr{E})$ to this groupoid topos (3) defining $\mathscr{P}(\mathscr{E})$. Thus, we obtain a natural comparison map $\Pi(\mathscr{E}) \to \mathscr{P}(\mathscr{E})$.

Proposition 9.1. *Assume $\mathscr{E}^I \to \mathscr{E} \times \mathscr{E}$ locally connected, as above. Then the natural comparison map $\Pi(\mathscr{E}) \to \mathscr{P}(\mathscr{E})$ is an equivalence of topoi.*

This proposition is an immediate consequence of the following general lemma, the proof of which is easy and omitted.

Lemma 9.2. *Let $f_\bullet : \mathscr{Y}_\bullet \to \mathscr{X}_\bullet$ be a map of simplicial topoi, and write $f : \mathscr{D}(\mathscr{Y}_\bullet) \to \mathscr{D}(\mathscr{X}_\bullet)$ for the induced map of descent topoi. If f_0 is an equivalence, f_1 is connected, and f_2 is a surjection, then f is an equivalence.*

The assumption of the proposition that $\mathscr{E}^I \to \mathscr{E} \times \mathscr{E}$ is locally connected is related to the property of $\mathscr{E}$ being locally path simply connected, used earlier:

Proposition 9.3. *For any l.p.s.c. topos $\mathscr{E}$, the map $\mathscr{E}^I \to \mathscr{E} \times \mathscr{E}$ is locally connected.*

Proof. The detailed proof is somewhat involved, and we only give an outline here.

First, the proposition holds when $\mathscr{E}$ is localic. Indeed, for a connected l.p.s.c. locale X one shows that $X^I \to X \times X$ is locally connected by a standard topological argument using the evident basis of X^I, given by chains of p.s.c. open sets in X.

Next, for a general topos $\mathscr{E}$, the lemma is reduced to the case of locales by using the cover $X = X_{\mathscr{E}} \xrightarrow{\varphi} \mathscr{E}$ from [7], which is connected and locally connected with contractible fibers. Then if $\mathscr{E}$ is l.p.s.c., the locale X will be l.p.s.c. as well. Thus, by the localic case, $X^I \to X \times X$ is a locally connected map. Now consider the square

$$\begin{array}{ccc} X^I & \xrightarrow{\varphi^I} & \mathscr{E}^I \\ {\scriptstyle \varepsilon'}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ X \times X & \xrightarrow[\varphi \times \varphi]{} & \mathscr{E} \times \mathscr{E}. \end{array}$$

By [12, 17], the map φ^I is an open surjection. It thus follows that ε is locally connected since $(\varphi \times \varphi)\varepsilon'$ is (see [13]). □

10. Concluding remarks

Example 10.1. The example of the long circle which, as a topological space has trivial path fundamental group yet a non-trivial Chevalley group, is mentioned in [1]. As pointed out in [18, 3], this "anomaly" disappears if one replaces the topological space X by its topos of sheaves. Indeed, X has a covering space $\tilde{X}$ that is connected and path simply connected, and this remains true for the corresponding topoi of sheaves. Since the conditions of Proposition 7.1 are satisfied, $\mathscr{E} = Sh(X)$ has the property that the basic comparison map $\varphi : \Pi(Sh(X)) \to SLC(Sh(X))$ is an equivalence. In particular, φ induces an isomorphism $\pi_1^{\text{paths}}(Sh(X), p) \cong \pi_1(Sh(X), p)$, for any chosen basepoint p, quite unlike the topological situation.

Remark 10.2. Let G be an étale complete localic groupoid such that its source and target maps $G_1 \rightrightarrows G_0$ are connected and locally connected, and let $\mathscr{B}G$ be its classifying topos. (Recall from [8] that every topos arises in this way.) By [17], the map $G_0^I \to (\mathscr{B}G)^I$ is an open surjection, hence an effective descent map. Thus, the exponential topos $(\mathscr{B}G)^I$ can be constructed as the classifying topos $\mathscr{B}(G^I)$ of the localic groupoid G^I. In particular, the path fundamental group of $\mathscr{B}G$ can be described in terms of paths in the locale G_0. In many concrete examples, this leads to an explicit description of the path fundamental group.

Remark 10.3. Let G be a topological groupoid such that its source and target maps are étale (local homeomorphisms). Then G is étale-complete [10] and $\mathscr{B}G$ is an étendue. Furthermore, $\mathscr{B}G$ is l.p.s.c. iff the space G_0 is locally (path) simply connected in the usual sense. In this case, our results imply that the Grothendieck fundamental group of $\mathscr{B}G$ can be described in terms of paths in $\mathscr{B}G$. These are "paths" α in G_0 with finitely many G-jumps, as

$$\alpha = (\alpha_0, g_1, \alpha_1, \ldots, \alpha_{n-2}, g_{n-1}, \alpha_{n-1}),$$

where $\alpha_i : [i/n, (i+1)/n] \to G_0$ is a continuous map (for $i = 0, \ldots, n-1$) and g_i is an arrow in the groupoid G from $\alpha_{i-1}(i/n)$ to $\alpha_i(i/n)$.

This applies in particular to the holonomy group $Hol(M,\mathscr{F})$ of a foliation $(M,\mathscr{F})$, and shows that the Van Est fundamental group of a foliation [21], which agrees (more or less by definition, see [16]) with the Grothendieck fundamental group of the classifying topos $\mathscr{B}(Hol(M,\mathscr{F}))$, can be described in terms of such "paths with jumps" in the holonomy groupoid. An explicit calculation will yield the description of the fundamental group of a foliation by paths discussed in [19, 20].

Remark 10.4. The various constructions of the "fundamental group" considered in previous sections all apply to a topos $\mathscr{E}$ defined over an arbitrary base topos $\mathscr{S}$, i.e., to a morphism $\gamma:\mathscr{E}\to\mathscr{S}$. If $\mathscr{E}$ is a l.p.s.c. topos over $\mathscr{S}$, all the constructions have been shown to agree. It follows that the constructions are *stable under change-of-base*, in the sense that for any map $f:\mathscr{S}'\to\mathscr{S}$, the canonical map

$$\Pi(\mathscr{E}\times_{\mathscr{S}}\mathscr{S}'\to\mathscr{S}')\to\Pi(\mathscr{E}\to\mathscr{S})\times_{\mathscr{S}}\mathscr{S}' \tag{4}$$

is an equivalence of topoi. Indeed, by Proposition 9.1, the topos $\Pi(\mathscr{E}\to\mathscr{S})$ can be constructed as the descent topos of a groupoid topos $\mathscr{H}\rightrightarrows\mathscr{E}$. This groupoid topos has the property that $\mathscr{H}\to\mathscr{E}\times\mathscr{E}$ is localic (in fact, a slice), so that this descent construction is pullback stable [14]. Since the construction of $\mathscr{H}$ itself is evidently stable as well, the claimed equivalence (4) follows.

Thus, for example, when $\gamma:\mathscr{E}\to\mathscr{S}$ has section $p:\mathscr{S}\to\mathscr{E}$, one obtains an isomorphism of groups in $\mathscr{S}'$,

$$f^*(\pi_1(\mathscr{E},p))\cong\pi_1(\mathscr{E}\times_{\mathscr{S}}\mathscr{S}',p'), \tag{5}$$

where p' is the evident section induced from p by f.

Question 10.5. Following the notation of the previous remark, is there a good formula for the composition of two l.p.s.c. morphisms $\mathscr{F}\to\mathscr{E}$ and $\mathscr{E}\to\mathscr{S}$, relating $\Pi(\mathscr{F}\to\mathscr{S})$ to $\Pi(\mathscr{F}\to\mathscr{E})$ and $\Pi(\mathscr{E}\to\mathscr{S})$, of the corresponding fundamental groups in the pointed case?

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